The ultraviolet-finite Hamiltonian approach on the noncommutative Minkowski space

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Abstract: Written version of a talk presented at the 36th International Symposium Ahrenshoop on the Theory of Elementary Particles, 26-30 August 2003, Wernsdorf, Germany.

This is an exposition of joint work with S. Doplicher, K. Fredenhagen, and G. Piacitelli on field theory on the noncommutative Minkowski space [1]. The limit of coinciding points is modified compared to ordinary field theory in a suitable way which allows for the definition of so-called regularized field monomials as interaction terms. Employing these in the Hamiltonian formalism results in an ultraviolet finite S-matrix.

1 Introduction

Noncommutative spacetimes are studied for various reasons, one of them being the thought experiment that Heisenberg's uncertainty relation in conjunction with the laws of classical gravity leads to a restriction as to the best possible localization of an event in spacetime. The idea is that the simultaneous measurement of two or more spacetime directions with an arbitrarily high precision requires an arbitrarily high energy which could result in building a horizon, cf. for instance [2]. Another motivation is based on string theory, where field theories on noncommutative spacetimes are derived as special low-energy limits of open string theories on D-brane configurations in background magnetic fields [3, 4].

The model on which our analysis is founded was defined in [2], where continuous spacetime is replaced by a noncommutative C^*-algebra $\mathcal{E}$ "generated" by Hermitean noncommutative coordinate operators $q^0, \ldots, q^3$ with $[q^\mu, q^\nu] = iQ^{\mu\nu}$, $\mu, \nu = 0, \ldots, 3$, subject to so-called "quantum conditions",

$$Q_{\mu\nu}Q^{\mu\nu} \;=\; 0\,, \qquad \left(\tfrac{1}{4}\, Q^{\mu\nu}\, Q^{\rho\sigma}\, \epsilon_{\mu\nu\rho\sigma}\right)^2 \;=\; \lambda_P^8\, I\,, \qquad [q^\rho, Q^{\mu\nu}] \;=\; 0 \tag{1}$$

where λ_P is the Planck length, I the identity (actually, the quantum coordinates, being unbounded operators, are affiliated to $\mathcal{E}$, see [2] for details). The quantum conditions are Poincaré invariant, and the commutators are not given by a fixed matrix. The joint spectrum Σ of the operators $Q^{\mu\nu}$ is homeomorphic to the non-compact manifold $TS^2 \times \{1, -1\}$.

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For any state ω in the domain of the $[q_\mu, q_\nu]$, the uncertainties $\Delta_\omega q_\mu = (\omega(q_\mu^2) - \omega(q_\mu)^2)^{-1/2}$ fulfill the following space-time uncertainty relations: $\Delta q_0 \cdot (\Delta q_1 + \Delta q_2 + \Delta q_3) \geq \lambda_P^2/2$, $\Delta q_1 \cdot \Delta q_2 + \Delta q_1 \cdot \Delta q_3 + \Delta q_2 \cdot \Delta q_3 \geq \lambda_P^2/2$. By a generalized Weyl correspondence in the spirit of ordinary quantum mechanics, the regular realizations of the quantum conditions were found in [2], the difficulty being the nontriviality of the spectrum of the commutators resulting in a nontrivial centre $\mathcal{Z}$ of (the multiplier algebra of) $\mathcal{E}$. In particular, the product of two "functions of the quantum coordinates" is given by the twisted convolution,

$$f(q)g(q) = \int dk\, dp\, \check{f}(k)\, \check{g}(p)\, e^{-\frac{i}{2}kQp}\, e^{i(k+p)q}\,, \qquad kQp = k_\mu Q^{\mu\nu} p_\nu\,,\ kq = k_\mu q^\mu\,, \tag{2}$$

with $f \in \mathcal{F}L^1(\mathbb{R}^4)$, $\check{f} = \mathcal{F}^{-1} f$, where $\mathcal{F}$ is the ordinary Fourier transform. $e^{-\frac{i}{2}kQp}$ is referred to as the twisting. The full Poincaré-group acts as automorphisms on $\mathcal{E}$ and derivatives may be defined as the infinitesimal generators of translations. The evaluation in a point $f(q) \to f(a)$, $a \in \mathbb{R}^4$, fails to be a positive functional. Instead, optimally localized states ω_a with localization centre $a \in \mathbb{R}^4$, minimizing the uncertainties, have been defined in [2]. Explicitly, for f as above and $g \in C_0(\Sigma)$,

$$\omega_a(g(Q)f(q)) = \int_{\Sigma_1} d\mu_\sigma g(\sigma) \int dk\ \check{f}(k)\, \omega_a(e^{ikq}) = \int_{\Sigma_1} d\mu_\sigma g(\sigma) \int dk\ \check{f}(k)\, e^{-\frac{1}{2}|k|^2}\, e^{ika}\,, \tag{3}$$

where $|k|^2 = {k_0}^2 + \cdots + {k_3}^2$ and where μ is any probability measure on a distinguished subset $\Sigma_1 \subset \Sigma$. The definition is rotation- and translation-invariant but not invariant under boosts. This is a problem in general, as states on $\mathcal{E}$ must take values in $\mathbb{C}$, and, therefore, one has to get rid of the dependence on the commutators by integrating with respect to some measure on Σ. Unfortunately, the Lorentz group is not amenable, and there is no obvious Lorentz-invariant measure on Σ. In analogy with the definition of $f(q)$, a free quantum field $\phi(q)$ on the noncommutative Minkowski space was defined in [2] as

$$\phi(q) \stackrel{\text{def}}{=} \int dk\ \check{\phi}(k) \otimes e^{ikq} = (2\pi)^{-3/2} \int \frac{d\,\mathbf{k}}{2\omega_{\mathbf{k}}} \left(a(k) \otimes e^{-ikq} + a^*(k) \otimes e^{ikq} \right) \big|_{k \in H_m^+} \tag{4}$$

with ordinary annihilation and creation operators a and a^*, $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$, and the ordinary positive mass-shell H_m^+. The field ϕ is to be interpreted as a linear map from states on $\mathcal{E}$ to smeared field operators, $\omega \mapsto \phi(\omega) = \langle I \otimes \omega,\ \phi(q) \rangle = \int dx\ \phi(x)\, \psi_\omega(x)$, where on the right-hand side, $\phi(x)$ is a quantum field on ordinary spacetime, smeared with a testfunction ψ_ω defined by $\check{\psi}_\omega(k) = \omega(e^{ikq})$.

In last year's talk I spoke about our results on the problem of unitarity [5], pointing out that a careful definition of the time-ordering leads to unitary perturbative setups for general noncommutative spacetimes, making the restriction to lightlike or space-space noncommutativity unnecessary. Two different unitary setups were discussed, the Hamiltonian approach already proposed in [2] and the Yang-Feldman approach, and it was pointed out in particular, that the internal lines in these unitary approaches, are not, in general, given by Feynman propagators (unless lightlike or space-space noncommutativity are assumed, in which cases the two unitary approaches coincide with the modified Feynman rules).

The interaction term employed was a normally ordered product $:\phi^n(q):$ as proposed in [2]. While this is a straightforward generalization of the ordinary local interaction term $:\phi^n(x):$, it is not the only possibility, and one of the important questions in the

field is how ordinary local interaction terms are to be replaced in the noncommutative setting. While plagued with problems such as the violation of causality, field theories on the noncommutative Minkowski space do allow for some notions of locality from which suitable interaction terms may be derived. One of the possibilities is a programme adapted to the Yang-Feldman approach resulting in the introduction of the so-called quasiplanar Wick products [6, 7]. Another possibility, which was elaborated in [1], is the topic of this talk. It is more natural in the Hamiltonian approach, and is based on re-defining the concept of products of fields evaluated at the same point. Since, by construction, strict localization on the noncommutative Minkowski space is impossible, it is argued that fields cannot be evaluated "at the same point", but only at points which are "close together". Using the optimally localized states, this notion is made precise and the so-called quantum diagonal map is introduced. Applying this map to define interaction terms, Hamilton operators are found which lead to ultraviolet finite theories for any ϕ^n-self-interaction, see [1]. An alternative proof is sketched here, see [7] for details.

2 An approximate limit of coinciding points

Starting point are mutually commuting sets of quantum coordinates, $q_j^\mu = I \otimes \cdots \otimes I \otimes q^\mu \otimes I \otimes \cdots \otimes I$, $j = 1, \ldots, n$ with q^μ in the j-th tensor factor. As in field theory on the ordinary Minkowski space, the product of fields at such different "points",

$$\phi(q_1) \ldots \phi(q_n) = (2\pi)^{-4n} \int dk_1 \ldots dk_n \, \check{\phi}(k_1) \ldots \check{\phi}(k_n) \, e^{ik_1 q_1} \ldots e^{ik_n q_n} \,,$$

is well-defined, mapping states to field operators on the ordinary Fock space. Starting from this expression, one may try to give meaning to a field monomial in coinciding points in order to define the generalization of a local interaction term. The exact limit of coinciding points cannot be assumed, since, contrary to the ordinary case, the relative coordinates $q_{ij}^\mu = q_i^\mu - q_j^\mu$ do not all commute with each other, making it impossible to set all differences to zero simultaneously. Instead, one may define an "approximate" limit of coinciding points [8, 9] and minimize the relative coordinates using the states with minimal uncertainty (optimal localization) with localization centre $a = 0$. For a related discussion see also [10]. A simplification used in the construction is to take the tensor product in the definition of the mutually commuting sets of coordinates not over the complex numbers $\mathbb{C}$ but over the centre $\mathcal{Z}$,

$$q_i^\mu \stackrel{\text{def}}{=} I \otimes_{\mathcal{Z}} \cdots \otimes_{\mathcal{Z}} I \otimes_{\mathcal{Z}} q^\mu \otimes_{\mathcal{Z}} I \otimes_{\mathcal{Z}} \cdots \otimes_{\mathcal{Z}} I \,, \tag{5}$$

which means, in particular, that the tensor product is linear with respect to twistings. The quantum coordinates q_j then satisfy the canonical commutation relations,

$$[q_j^\mu, q_k^\nu] = 0 \text{ for } j \neq k \,, \qquad [q_j^\mu, q_j^\nu] = i \, Q^{\mu\nu}$$

where the right-hand side does not depend on j, such that, in particular, differences of commutators $[q_j^\mu, q_j^\nu] - [q_k^\mu, q_k^\nu]$ are zero. Here, the $Q^{\mu\nu}$ are subject to the quantum conditions (1). Employing the tensor product over $\mathcal{Z}$ also implies that the mean coordinates $\bar{q}^\mu = \frac{1}{n}(q_1^\mu + \cdots + q_n^\mu)$ commute with the relative coordinates, and behave like quantum coordinates of characteristic length $1/\sqrt{n}$, i.e. $[\bar{q}^\mu, \bar{q}^\nu] = i \, \frac{1}{n} Q^{\mu\nu}$.

Each quantum coordinate q_j^μ is then rewritten in terms of the mean coordinate and the relative coordinates, and a so-called quantum diagonal map $E^{(n)}$ is be defined, which

minimizes all relative coordinates using the states of optimal localization, while leaving the mean coordinate invariant. For the field monomial we then find explicitly, cf. [1]:

$$\phi_R^n(\mathfrak{q}) \stackrel{\text{def}}{=} E^{(n)}(\phi(q_1)\dots\phi(q_n)) = \int dk_1\dots dk_n\; \check{\phi}(k_1)\dots\check{\phi}(k_n)\, r_n(k_1,\dots,k_n)\, e^{i(\sum_i k_i)\mathfrak{q}}\,,$$

where the quantum coordinates $\mathfrak{q}^\mu$ with characteristic length $1/\sqrt{n}$ correspond to the mean coordinates, and where the kernel r_n is given by

$$r_n(k_1,\dots,k_n) = \exp\big(-\tfrac{1}{2}\sum_{i=1}^n \big|k_i - \tfrac{1}{n}\sum_{l=1}^n k_l\big|^2\big)\,. \tag{6}$$

The Gaussian factors result from the application of the states of optimal localization to the relative coordinates. Note that no twistings appear due to the fact that we started from mutually commuting coordinates[1].

3 Dyson's series

This definition is now applied in an effective perturbation theory on the ordinary Minkowski space based on the ordinary Dyson series and an effective nonlocal Hamilton operator. Starting point is the symbol $\phi_R^n(x)$ of $\phi_R^n(\mathfrak{q})$, $\phi_R^n(\mathfrak{q}) = (2\pi)^{-4}\int dk\; e^{ik\mathfrak{q}}\int dx\, e^{-ikx}\,\phi_R^n(x)$, for which we find

$$\phi_R^n(x) = c_n\int da_1\dots da_n\,\exp\big(-\tfrac{1}{2}\sum_{j=1}^n|a_j-x|^2\big)\,\delta^{(4)}\big(\sum_{j=1}^n a_j - n\,x\big)\,\phi(a_1)\cdots\phi(a_n)\,. \tag{7}$$

Due to the Gaussian functions, the operator valued distribution $\phi_R^n(x)$ may be called a *regularized field monomial*, since the evaluation in a testfunction $g\in\mathcal{S}(\mathbb{R}^4)$,

$$\int dx\, g(x)\,\phi_R^n(x) = \int da_1\dots da_n\; g(\tfrac{1}{n}\sum_{i=1}^n a_i)\,\prod_{j=1}^n \exp\big(-\tfrac{1}{2}|a_j - \tfrac{1}{n}\sum_{i=1}^n a_i|^2\big)\;\phi(a_1)\dots\phi(a_n)\,,$$

is well-defined. This follows directly from the fact that the product of g and the Gaussian functions as above provides a testfunction on $\mathbb{R}^{4n}$, and that the tensor product of fields is always well-defined. Note that the Gaussian functions alone do not yield a testfunction on $\mathbb{R}^{4n}$ as the arguments are not linearly independent, their sum being zero. For details see [7]. In other words, there is no need to bring the annihilation and creation operators in the regularized field monomial $\phi_R^n(x)$ into normal order as in the ordinary case, and all tadpoles turn out to be finite. Nevertheless, the effective Hamiltonian is defined with a normally ordered interaction,

$$H_I^g(t) = \tfrac{1}{n!}\int dx\,\delta(x_0-t)\,g(x)\,{:}\phi_R^n(x){:}\,, \tag{8}$$

where the regularized Wick monomial ${:}\phi_R^n(x){:}$ is defined as in (7), but with a normally ordered tensor product of fields, ${:}\phi(a_1)\cdots\phi(a_n){:}$. Here, an adiabatic switching $g\in\mathcal{S}(\mathbb{R}^4)$, which does not have a counterpart on the noncommutative side, is applied to postpone questions concerning the infrared-behaviour of the theory. Using the regularized field

[1] To be exact, the reader is reminded that the application of states of optimal localization also involves an integration with respect to some measure on Σ_1. However, $E^{(n)}(\phi(q_1)\dots\phi(q_n))$ is constant as a function of $\sigma\in\Sigma_1$ and no explicit dependence on $\sigma\in\Sigma_1$ appears.

monomials also in the definition of the free Hamiltonian density would result in a modified Hamilton operator which is no longer the zero component of a Lorentz vector. Hence, the free part of the theory and the interaction term are treated on different footings, and the former is assumed to be governed by the free Hamiltonian on the noncommutative Minkowski space introduced in [2], which turned out to be the same as the one on the ordinary Minkowski space. The free part of the theory requiring normal ordering makes it natural to assume the same for the interaction term. For a general discussion of the Hamiltonian approach on noncommutative spacetimes, including a discussion of the resulting graph theory and its equivalence to the so-called interaction point time ordering approach, see [7].

Following the proposal from [2], the S-matrix corresponding to the interaction Hamilton operator $H_I^g(t)$ was defined in [1], employing the ordinary Dyson series, $S[g] = I + \sum_{r=1}^{\infty} S_r[g]$, where

$$S_r[g] \;=\; \big(\tfrac{-i}{n!}\big)^r \int dt_1 \ldots dt_r\, \theta(t_1 - t_2) \ldots \theta(t_{r-1} - t_r)\, H_I^g(t_1) \ldots H_I^g(t_r)\,. \tag{9}$$

Note that S is formally unitary by the fact that the effective Hamiltonian is symmetric, $H_I^g(t)^* = H_I^g(t)$. Moreover, as we have shown in [1], due to the regularizing Gaussian kernel, the S-matrix $S[g]$ is well-defined in every order r. No ultraviolet divergences appear.

As an alternative to the analysis in [1], a proof in position space can be given, see [7]. The idea is the following. First consider ordinary quantum field theory, where the S-matrix at r-th order is given by

$$S_r[g] = c \int dx^1 \cdots dx^r\; g(x^1) \ldots g(x^r) \prod_{j=1}^{r-1} \theta\big(x_0^j - x_0^{j+1}\big) \prod_{j=1}^{r} {:}\phi(x^j)^n{:}\,.$$

Expectation values in multi-particle states (without smearing in the momenta) are typically of the form $\prod_{j<j'} \Delta_+(x^j - x^{j'})^{n(j,j')}\, \langle p_{(l)}| {:}\phi(x^1)^{m_1} \ldots \phi(x^r)^{m_r}{:} |q_{(s)}\rangle$ for some choice of indices $j, j' \in R = \{1, \ldots, r\}$. Here, $n(j,j') = n(j',j) \in \mathbb{N}_0$ and $m_j = n - \sum_{i \in R} n(j,i)$. While the multiplication of a translation-invariant distribution with a Wick product of fields is well-defined, ultraviolet divergences arise since the product of a Heaviside function θ with contractions Δ_+^n is ill-defined in 0 for $n \geq 2$. Such divergences do not appear if regularized Wick monomials are employed. Here, we find the following expression for $S_r[g]$:

$$\begin{aligned} S_r[g] \;=\; & c \int d\underline{a}^1 \cdots d\underline{a}^r\; g(\kappa(\underline{a}^1)) \ldots g(\kappa(\underline{a}^r)) \prod_{j=1}^{r-1} \theta\big(\kappa_0(\underline{a}^j) - \kappa_0(\underline{a}^{j+1})\big) \\ & \cdot \prod_{j=1}^{r} \Big(\;\exp\big(-\tfrac{1}{2}|\underline{a}^j - \kappa(\underline{a}^j)|^2\big)\; {:}\phi(a_1^j) \ldots \phi(a_n^j){:}\; \Big)\,, \end{aligned} \tag{10}$$

where $\underline{a} = (a_1, \ldots, a_n)$, $\underline{a} - x = (a_1 - x, a_2 - x, \cdots, a_n - x)$, $d\underline{a} = da_1 \ldots da_n$ for $x, a_j \in \mathbb{R}^4$, and where $\kappa(\underline{a}) \in \mathbb{R}^4$ is the mean of $\underline{a}$, $\kappa(\underline{a}) = \frac{1}{n}(a_1 + \cdots + a_n)$, and has the time component $\kappa_0(\underline{a}) = \frac{1}{n}(a_{1,0} + \cdots + a_{n,0})$. Taking expectation values of $S_r[g]$ in multi-particle states (without smearing in the momenta) then yields contractions which appear *exactly once*. Hence, multiplication with the Heaviside functions does not pose a problem. Moreover, as the product of Heaviside functions in (10) is a translation-invariant distribution, its multiplication with the Wick product of fields is not problematic. And as the arguments of

the contractions differ from those appearing in the fields, the product of contractions and fields is actually a tensor product and as such automatically well-defined. Furthermore, by a similar argument as employed in the discussion of the regularized field monomials, one concludes that the Gaussian functions together with the r adiabatic switching functions at r-th order yield a testfunction on $\mathbb{R}^{4nr}$. This proves the claim.

If no normal ordering is employed in the definition of the interaction Hamiltonian, the S-matrix is still finite at every order of the perturbative expansion as long as an adiabatic switching g is employed. An analysis of the adiabtic limit where g is replaced by a constant may be found in [7]. For instance, the vacuum expectation values of the S-matrix (9) diverge in this case. Furthermore, certain tadpoles turn out to diverge which provides further motivation as to why to consider the normally ordered interaction term. Moreover, the spatial cutoff may safely be removed in non-vacuum graphs, if the normally ordered interaction term is applied. Trying to remove the time-cutoff, however, one encounters a peculiar kind of divergence in particular graphs which are not one-particle irreducible. Such divergences will be absent when dressed propagators are employed - for which there is no need from the point of view of ultraviolet divergences. This may be seen as a subtle form of the ultraviolet-infrared mixing problem which otherwise is not present in the approach presented here, since the Gaussian kernels never cancel. For more details see [7].

4 Conclusion

The regularized Wick products can also be applied to theories with constant noncommutativity matrix θ, yielding an ultraviolet finite S-matrix for θ with maximal rank. However, if θ is degenerate as is the case in theories with commuting time, the best-localized states may not suffice as a regularization. Major drawbacks of the approach presented here (apart from fundamental problems such as the well-posedness of the initial value problem) are that the definition of the best localized states and hence that of the quantum diagonal map breaks Lorentz covariance, although rotation and translation invariance are kept, and that the free part of the theory is treated on a different footing than the interaction term. Nonetheless, the regularized Wick monomials do provide a framework in which one of the original aims of noncommutative spacetimes, namely to yield ultraviolet finite field theories, can be achieved.

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